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Application of derivative maps to Homotopy

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Abstract

The perspective of unification of mathematical concepts of Group Actions and Homotopy have been the bottom line of our study. We exploit this to a higher degree by investigating the derivatives associated with these. Heading in this direction, this paper is about linking the derivative of Homotopy and the derivative of Group Action. Firstly, we verify if the derivative of a Group Action is itself a Group action and whether the derivative of a Homotopy is a Homotopy. For this purpose, we take only special Group Actions and Homotopies restricted to the Euclidean space. We then discuss when the derivative of Group Action is a Homotopy and vice-versa. Thus our aim here is to find if the derivative of a Homotopy can lead to the existence of a related Group Action and the relevant criteria that must be satisfied for such a relation.

This paper also investigates when the derivative of Homotopy between two functions is a Homotopy in addition to being a Group Action as well. The derivative of Group Action and Homotopy are dealt with in an attempt to find if the derivative of Group Action is also a Homotopy. Since the derivative of a special action is also an action, we verify if this action is a Homotopy. Thus we interlink the derivative of the concepts of our study as theorems/propositions. In particular, we obtain conditions for the derivative of a Homotopy to be a Group Action.

Summarizing, this paper is about finding the derivative of Group Action and Homotopy if the presence of one leads to the existence of the other and if so when.

1 INTRODUCTION

This study deals with the advanced mathematical concepts of Group Action and Homotopy. In layman's language one may describe a 'Group Action' as the Action of one set on another set which alters the elements of the latter from within the second set. While 'Homotopy' is the continuous deformation of one curve into another curve between two fixed points (3). What appears to be interesting is the possibility of a Group Action being moved along a Homotopy. This leads to the question if the two concepts co-exist in the first place. Since they do co-exist on certain conditions determined in our previous work, we investigate if the derivatives of the associated concepts are interlinked and if so, under what conditions.

The usual derivative of two Homotopic maps need not be Homotopic. Interpreting Homotopy as a mapping between two Euclidean spaces allows us to define the derivative map (1) of a Homotopy and

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to show that this derivative map is a Homotopy between f ' and g '. In this paper, we investigate when the derivative of Homotopy between two functions is a Homotopy. For this purpose, we consider the Euclidean space E^3 and interpret Homotopy of two curves in E^3 to be a map on E^3. Thus we define a new derivative of Homotopy called the derivative map of the Homotopy. This paper also investigates as to when the derivative of Homotopy between two functions is a Group Action in addition to being a Homotopy as well.

The derivative of Group Action and Homotopy are dealt with an attempt to find if derivative of group action is also a Homotopy and vice versa. Since the derivative of a special action on E^3 is also an action, we verify if this action is a Homotopy as well. Thus we interlink the derivative of the concepts of our study with the help of some theorems/propositions on these. This leads to adjoining this to generalization and unification approach (4). Our aim is to obtain conditions for derivative of a Homotopy to be a group Action and for the derivative of Group Action to be a Homotopy as well. As an application of the chosen concepts of our study we would like to investigate piecewise Homotopy (not to be confused with piecewise Homotopy analysis) in our future work. In connection with geometric topology, several problems can be dealt offering a rich structural setting by the piecewise linear Homotopy.

This paper contains a section on Introduction, section 2 on preliminaries followed by section 3 comprising of main results obtained. Section 4 deals with the conclusion.

2 PRELIMINARIES

Definition 2.1: Group Action

A group (G, f) is said to act on a set X if $\exists$ a function F: G $\times$ X $\to$ X such that

(i) F (e, x) = x and (ii) F(f(g_1, g_2), x) = F (g_1, F(g_2x))

We have the convention that g(x) = F(g, x) i.e., 'g' acts on x which we again write as gx

Definition 2.2: Homotopy

The mappings f, g: X $\to$ Y are said to be homotopic if and only if there exists a continuous map F: X $\times$ (0, 1) $\to$ Y such that the following conditions are satisfied

(i) F(x, 0) = f(x) and (ii) F(x, 1) = g(x), $\forall$ x. Then f $\simeq$ g.

Consider the homotopy F: $E^3\times$ I $\to$ E^3, for each $t\ \in [0,\ \ 1]$, F$_t$: $E^3\to E^3$. This interpretation allows us to define the derivative map F$_{t*}$ which carries a tangent vector to a straight line to a tangent vector of the curve which is the image of a straight line at F$_t$ (p).

Definition 2.3: Set-Homotopy

Let f, g: X $\to$ Y. We say that the functions f, g are called set-homotopic or projection of sets if $\exists$ a mapping F: X $\times$ Z $\to$ Y where Z is some set (it could be a topological space or a group, etc.) such that F(x, a) = f(x) and F(x, b) = g(x) where 'a' and 'b' are two fixed points of Z.

This definition is silent on the continuity of f, g, and F because X, Y, and Z are general sets and continuity cannot be defined on these.

Definition 2.4: Derivative map

Let F: $E^n\to E^m$ be a mapping. If v is a tangent vector to E^n at p, let $F_*(v)$ be the initial velocity of the curve t $\to$ F(p + t v) in E^m. The resulting function F_* (from tangent vectors of E^n to tangent vectors of E^m) is called the derivative map of F. Thus for each point ‘p’ in E^n, the derivative map F_* gives rise to a function F_{*p} :$T_p\,(E^n) \to T_{F(p)}\,(E^m)$, which is called the derivative map of F at point p.

Definition 2.5: Derivative of Group Action

If F: $E^3\times\, E^3 \to E^3$ is an action then the derivative of F is defined by G = F $\left(\frac{dv}{dt},\ p\right)$
$\therefore$ G’ = F $\left(\frac{d^2v}{dt^2}\ ,\ \ p\right)$; v $\in E^3$, $p\ \in\ E^3$

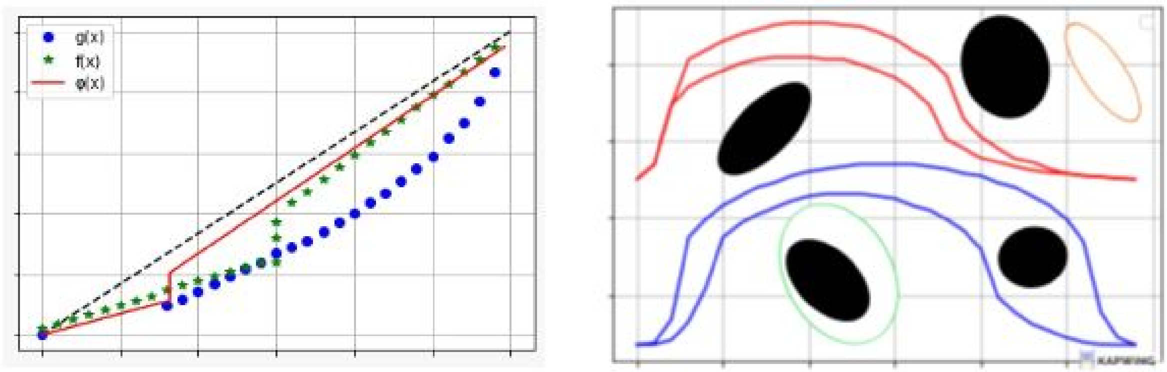
Definition 2.6: Derivative of Homotopy

The derivative of Homotopy F_t: $E^3\times$ (0, 1) $\to E^3$ between two functions f and g is defined as the derivative map of F_t : $E^3\!\to E^3$ i.e., if F_t: $E^3\times$ (0, 1) $\to E^3$ is a Homotopy between f and g then the derivative of F, D_F is defined by the map $D_F\ =\ F_{t*}\ :\ E^3\!\to E^3$ such that $D_F\ =\ \frac{d}{dt}\ (F_t(p+tv))]$ for ; $t\ \in [0,\ \ 1]$

We show that the derivative of an action of a group is also a group action and the derivative map of a Homotopy between f and g is a Homotopy between f ’ and g ’ in the following theorems under “Main Results”. The derivative of the Homotopy, F_{t*} (v, t) = $\frac{d}{dt}\ (F_t(p+tv))]\ |_{t=0}\ ;\ \ t\ \in [0,\ \ 1]$ is a Homotopy such that the following conditions are satisfied

Definition 2.7: Piecewise Homotopy

Let f, g: X $\to$ Y be continuous maps with Y being a subset of the Euclidean space with the subspace topology. Suppose there exists continuous maps $f_0,\ f_1,\ f_2,\ f_3,\ldots\ f_n$: X$\to$ Y such that f= $f_0$and g = f_n and linear homotopies $F_{i(i+1)}$ between each f_i and f_{i+1}. Then we define a composite of Homotopies $F_{i(i+1)}$ which is a homotopy from f to g. A Homotopy obtained in this way is termed as Piecewise Linear Homotopy.



(a) Piecewise linear (b) Homotopic strategies

FIGURE 1

Schematic representation ofPiecewise Homotopy

3 MAIN RESULTS

In this section we deal with few results pertaining to the Derivative of Group Action and Derivative of Homotopy. We shall incorporate the criteria that the group actions considered here must be differentiable maps.

Theorem 3.1:

If F: $E^3 \times E^3 \to E^3$ is an action then the derivative of Group Action is also a Group Action.

Proof: Let F be a Group Action and G be its derivative then G = F$\left(\frac{dv}{dt},\ p\right)$, is also a group Action, where v is any vector on E^3 and p is any point on E^3. For G to be a Group Action, this function must satisfy the conditions:

(i) G(e, x) = x , $\forall$x $\in$ G

(ii) G(f(g_1, g_2), x) = G(g_1, G(g_2, x))

In terms of F we can rewrite these as:

F($\frac{de}{dt}$, x) = $\frac{de}{dt}$ + x = 0+x = x

F($\frac{d\ (g_1 +\ g_2\ )}{dt}$, x) = F($\frac{dg_1}{dt}$ + $\frac{dg_2}{dt}$, x)

= F $\left(\frac{dg_1}{dt}\right)$, F$\left(\left(\left(\frac{dg_2}{dt},\ x\right)\right)\right)$

Since F is a Group Action itself, it necessarily satisfies the condition. Hence G= F$\left(\frac{dv}{dt},\ p\right)$ is also a Group Action.

Theorem 3.2:

If f, g: $E^3 \to E^3$ are continuous and map F is the Homotopy between them, then the derivative map F_X is also a Homotopy between $f^{'}$ and $g^{'}$.

Proof: Let f $\simeq$ g. Hence $\exists$ F: $E^3 \times$ I $\to E^3$ such that F is continuous and F(v, 0) = f(v) and F(v, 1)= g(v), v $\in E^3$. For each s $\in$ I, F_s: $E^3 \to E^3$is continuous. $\therefore$ we can consider the derivative F_{s*} defined as:

F_{0*} = $\frac{d\ F_s\ (p+\ tv)}{dt}|_{t=0}$; t $\in$ [0, 1] such that

F_{0*} = $\frac{d\ F_0\ (p+\ t(0))}{dt}|_{t=0}$

= $\frac{d\ F(p+\ tv,\ \ 0)}{dt}|_{t=0}$

= $\frac{d\ F(p,\ \ 0)}{dt}|_{t=0}$

= f ’(s) at p.

F_{1*} (v) = g’(s) at p. Hence f ’ $\simeq$ g ’.

Thus the derivative map of a Homotopy is also a Homotopy.

Remark 3.1 : The derivative map of a Homotopy is a translation applied at v and its variation for different values of t $\in$ [0, 1] WHILE the derivative of a Group Action is the derivative at point v and then the translation by values of t $\in$ [0, 1].

Our next theorem is about obtaining a condition for the derivative of a group action to be a Homotopy.

Proposition 3.3:

Proof: Let γ: $E^3 \times E^3 \to E^3$ be a Group Action.

If γ is a Group Action and $\gamma\ '$ denotes its derivative, then $\gamma\ '$ is a Homotopy if the condition $\frac{du}{dt} = f(t)$ holds.

$\therefore \gamma$ satisfies the conditions (i) γ (0, $\overrightarrow{u}$,) = $\overrightarrow{u}$ and (ii) γ (f($\overrightarrow{v_1}$ + $\overrightarrow{v_2}$), $\overrightarrow{u}$) $= \gamma$[$\overrightarrow{v_1}$, γ ($\overrightarrow{v_2}$, $\overrightarrow{u}$)]

Let $\gamma^{'}$ be the derivative of Group Action. Then by definition we have $\gamma^{'}\ (\overrightarrow{u},\ v) = \gamma(\ \frac{du}{dt}\ ,\ v)$

Consider $\gamma^{'}\ (u,\ 0) = \gamma\ \left(\ \frac{du}{dt}\ ,\ \ 0\right)$ = f($\Rightarrow \frac{du}{dt} =\ f\ (t)$

For the purpose of relating the derivative of Group Action to Homotopy, we restrict E^3 to [0, 1]. A point '1' of the interval [0, 1] in E^3 means fixed points.

Also F(x, 1) = g(x)

i.e., $F'(u,\ 1)\left(\frac{du}{dt}\ ,\ \ 1\right)$ = g($\overrightarrow{u}=\ F$)

$\therefore \left(\frac{du}{dt}+\ \ 1\right)$ = g($\overrightarrow{u}$) where '1' is a vector or a fixed point.

$\Rightarrow$ f(u) + 1 = g(u) i.e., F(u) – g(u) = 1, a fixed point.

Thus the derivative of a Group Action is a Homotopy.

Proposition 3.4:

The derivative of a Homotopy between the two functions f and g, is a Group Action.

Proof: The derivative map of Homotopy is considered as the derivative map of $F_t\ : E^3\ \rightarrow E^3,\ \ \forall t\ \in$ $(0,\ 1]$ defined as:

$F_{t*}\ (\overrightarrow{v_p})\ \ =\ \frac{dF_t\ (p+tv)}{dt}\ |_{t=0}$

i.e., the tangent on the curve is mapped to the image in E^3 with the help of this derivative map F_{t*} of the Homotopy F_t . We need to show that this map F_{t*} is a Group Action. i.e. we must show that the below conditions are satisfied (i) $F_{t*}\ (\overrightarrow{v_p},\ \ p)$ = p and (ii) F_{t*} ($\overrightarrow{v_p+\ w_p}$, p) = F_{t*} ($\overrightarrow{v_p}$, F_{t*} ($\overrightarrow{w_p}$, p))

Consider the Homotopy F: $E^3\times\ E^3\rightarrow E^3$ rewritten as the mapping $F_t\ :E^3\ \ \rightarrow E^3,\ \ \forall t\ \in[0,\ 1]$ defined as:

$F_{t*}\ (\overrightarrow{v_p})\ \ =\ \frac{dF_t\ (p+tv)}{dt}\ |_{t=0}$ where we take the same t throughout.

To show that this derivative map is F_{t*} is a group action, this map must satisfy the below conditions:

(i) $F_{t*}\ (e,\ \ \overrightarrow{v_p}\ ) = \overrightarrow{v_p}$ and (ii) F_{t*} ($\overrightarrow{v_p+\ w_p}$, p) = F_{t*} ($\overrightarrow{v_p}$, F_{t*} ($\overrightarrow{w_p}$, p))

(i) Since $F_{t*}\ (e,\ \ \overrightarrow{v_p}\ )= \frac{dF_t\ (e+tv)}{dt}\ |_{t=0} = \frac{dF_t\ (tv)}{dt}\ |_{t=0}\ = \overrightarrow{v_p}$

(ii) Consider the L.H.S F_{t*} ($\overrightarrow{v_p+\ w_p}$, p) being considered at the point 'p'

F_{t*} ($\overrightarrow{v_p+\ w_p}$, p) = $F_{t*}\left(\overrightarrow{v_p+\ w_p},\ p\right) = \frac{dF_t\ (p+t(v+w))}{dt}\ |_{t=0} = \frac{dF_t\ (p+tv+tw)}{dt}\ |_{t=0}\ \ ---\mathbf{1}$

Consider the R.H.S F_{t*} ($\overrightarrow{v_p}$, F_{t*} ($\overrightarrow{w_p}$, p)) = F_{t*} ($\overrightarrow{v_p}$, $\frac{dF_t\ (p+tw)}{dt}\ |_{t=0}$)

$= \frac{dF_t\ (\ (p+tw)\ +\ tv)}{dt}\ |_{t=0}$

$= \frac{dF_t\ (\ p+tw\ +\ tv)}{dt}\ |_{t=0}\ \ \ ----2$

From (1) and (2) above, the derivative map of a Homotopy is a Group Action. The derivative map transforms a tangent vector to a curve in E^3, each point p_i comprises of the components $(p_1,\ \ p_2,\ \ p_3)$.

We know that a Homotopy transforms one curve into another continuously and hence it transforms a straight line into a curve continuously as well which shows intuitionally that a tangent to the straight line is transformed to the corresponding curve. This is now proved as a theorem.

Homotopy as a Derivative Map:

Theorem 3.5: Any Homotopy on E^3 is also a derivative map.

Proof: Homotopy is basically transforming one curve into another. $\therefore$ a tangent to one curve at a point should be transformed to the tangent by the other curve.

We conjecture that it is possible to represent a Homotopy as a derivative map.

4 Application and Future work

The novelty of this paper is mainly to establish the connection between the derivative of homotopy and derivative of group action. Further to this, in order to give an eye-view of the associated application linking the mathematical concepts of Homotopy, group action and their derivatives, we first work at establishing homotopy and group action in cochlear functioning. Further the resulting bio-mathematical application would serve as the basis for finding a connection of the results obtained in this paper. Just to help us understand its connection we include a part of the association that helps to connect to health and well-being associated with the human ear functioning.

Anatomy of neuron: An action potential is an explosion of depolarizing current that travels along the cell. For an action potential to occur, the depolarization must reach a minimum threshold voltage. Action potentials are only fired as an 'all -or-none" response. This means that action potentials do not varying in size and will not occur if the threshold is not reached.

The electrical activity can be summarized as the famous Hodgkin-Huxley model represented by:

$C_m \frac{dV_m}{dt} + I_{ion} = I_{ext}$

Where $C_m \rightarrow$ $membrane\ capacitance$

$V_m \rightarrow$ $intracellular\ potential$

t $\rightarrow$ time

$I_{ion} \rightarrow$ $net\ ionic\ current$

$I_{ext} \rightarrow$ $externally\ applied\ current$

Most significant is to determine the membrane capacitance in such a way that it is independent of the sign or magnitude of the intracellular potential and minimally affected by the time course of V_m . (11)

To develop the differential equations that describe the conductance, the probability of a gate being open is defined as p_i for any ion 'i'.

Also $p_i \rightarrow fraction\ of\ gates\ in\ the\ permissive\ state.$

$$1 - p_i \rightarrow amount\ of\ gates\ closed.$$

To obey the first order kinetics, $\frac{dp_i}{dt} = \alpha_i\ (v)\,(1 - p_i\,) - \beta_i\,(v)\ p_i$,

Where α_i , $\beta_i \rightarrow$ rate constants dependent on voltage describing transient rates of permissive and non-permissive gates (12).

5 Application of the result -Derivate of a group Action is a Homotopy

The group action resulting from the vibrations of BM acts in the form of movement of hair cells(deflection of stereocilia) resulting in the polarization / depolarization of hair cells.

Considering the group action as the mapping $F_1 : BM \times HC \rightarrow HC$ defined as

$F_1\ (x \in BM,\ y \in HC) = y_{P/D}$ where polarization / depolarization of hair cells. This gives us the net amount of current added or received from each neuron.

The derivative of this group action is defined as $F_1{}^*$: $BM \times X \rightarrow X$, where X represents the set of hair cells, may be defined as $F_1{}^*\,(x \in BM,\ y \in HC) = y_{P/D}$. The derivative of a group action is also a group action. Further this derivative is a homotopy, associated to the rate of change of flow of current which happens continuously. This derivative $F_1{}^*$ gives the rate of acceleration of the increase / decrease of current flow as a product.

This is the idea behind associating derivatives within the Basilar membrane functioning in the cochlea. A mathematical expression will be attempted preferably relating to an existing well-established mathematical model, in the future.

6 CONCLUSION

This research paper deals with the derivative map of Homotopy and its link to Group Action. With a view to establish a unified theory, we look at various possibilities where the presence of one admits the other and how it relates to the derivative of the other. The concepts of Group Action and Homotopy are interlinked under special cases. These mathematical concepts help to establish a connection to health and well-being which we intend to explore in the future. The novelty of this paper is mainly to establish the connection between the derivative of homotopy and derivative of group action. In the associated application linking the derivatives of the chosen mathematical concepts of Homotopy, group action, we get a glimpse of the feasible application of the results obtained in this paper. Thus, mathematically connecting to health and well-being associated with the human ear functioning.

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